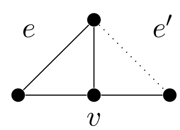
Chapter 1 - Connectivity

Let G be a graph and $v \in V(G)$, $e \in E(G)$, and $e' \in \binom{V}{2} \setminus E(G)$

- $G - e = (V, E \setminus \{e\})$ is **edge deletion**
- $G + e' = (V, E \cup \{e\})$ is a new **edge addition**
- $G - v = (V \setminus \{v\}, \{e \in E, v \notin e\}$ is **vertex deletion**
- $G\%e = (V \cup \{z\}, (E \setminus \{e\}) \cup \{\{z, x\}, \{z, y\}\})$, where $e = xy$ and z is a new vertex is **edge subdivision**

1: Perform $G - e$, $G + e'$, $G - v$, and $G\%e$ on the following graph G.



Solution:

A graph G is **connected** if for every two vertices u, v, the exists a walk in G from u to v.
A **connected component** in G is a maximal subgraph of G that is connected.

Complement of a complete graph is an **independent set**.

2: Show that if G is disconnected then $\overline{G}$ is connected.

Solution: Let $V(G) = V_1 \cup V_2$, such that there are no edges between V_1 and V_2 and each is non-empty. Now verify the definition that $\overline{G}$ is now connected.

A **seperating** set of G is a set of vertices or edges, S, such that the number of components of G is less than the number of components of $G - S$.

Vertex v is a **cut vertex** of G if $\{v\}$ is a separating set of G. Edge e is a **bridge** if $\{e\}$ is a separating set of G.

3: Find a graph that has more cut-vertices than bridges and a graph that has more bridges than cut-vertices.

Solution: A star has more bridges than cut-vertices, or any tree. Two triangles sharing a vertex have one cut-vertex but bridges.

A graph G is called k**-connected** (for $k \in \mathbb{N}$) if $|G| > k$ and $G - X$ is connected for every set $X \subseteq V$ with $|X| < k$. Largest k such that G is k-connected is called **connectivity** of G, denoted $\kappa(G)$.

Notice $\kappa(K_n) = n - 1$.

A graph G is called ℓ**-edge** if $G - X$ is connected for every set $X \subseteq E$ with $|X| < \ell$. Largest ℓ such that G is ℓ-connected is called **edge-connectivity** of G, denoted $\lambda(G)$.

4: Show that if G is non-trivial then $\kappa(G) \le \lambda(G) \le \delta(G)$.
High connectivity requires high minimum degree.

Solution: Second inequality holds because edges incident to a minimum degree vertex form an edge-cut.

Let F be an edge-cut with $|F| = \lambda(G)$. If exists v not incident to any edge in F, we take endpoints of F from it's side of cut. So every vertex is adjacent to something in F. Notice that $|N(v)| \le |F|$. We take cut $G - N(v)$. If $G - N(v) = v$ for all vertices v, we have a complete graphs and $\kappa(G) = \lambda(G) = |G| - 1$.

5: Which of these 2 graphs is connected? Identify connected components.

Solution: Left one is, right one is not.

6: Find a complement of the following graphs.